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Performance Analysis of a Preemptive Priority Queue with Applications to Packet Communication Systems

By M. G. HLUCHYJ,* C. D. TSAO,* and R. R. BOORSTYN†

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In this paper we analyze the performance of a preemptive priority queue. We give the model description in the context of a packet communication system where message sources, having different priorities, share a common communication channel. Each source generates, as an independent Poisson process, messages consisting of an arbitrarily distributed, random number of fixed-length packets. The channel server can only begin service at integer multiples of the packet transmission time (i.e., a time-slotted channel is assumed), and the server will preempt an ongoing message transmission at the next packet boundary whenever there is a message arrival from a higher-priority source. The average in-queue waiting time for each packet in any given source message and the average message delay are derived along with the corresponding moment-generating functions. Also, comparisons are made with the first-come first-served queueing discipline.

I. INTRODUCTION

We analyze the performance of a preemptive priority queueing system. To make clear at the outset the importance of the particular queueing system studied, we describe the system model in a packet communication context. Specifically, as Fig. 1 illustrates, a number of data sources share a single communication channel. Each source generates, according to a Poisson process, messages consisting of a

*AT&T Information Systems. †Polytechnic Institute of New York.

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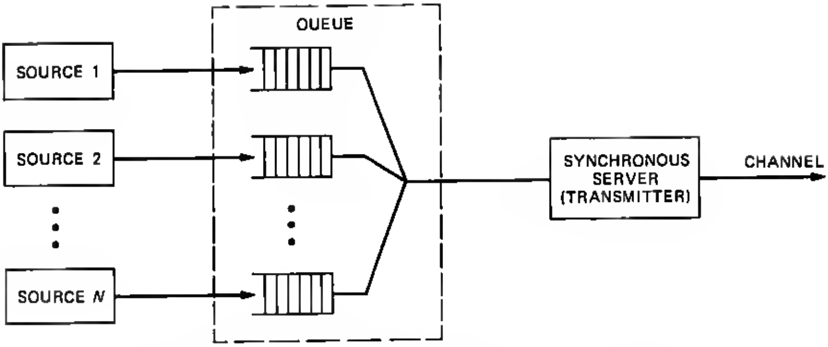

Fig. 1—Queueing model for a packet communication system.

random number of fixed-length data packets. The packets comprising a message arrive in bulk to be transmitted on the communication channel. For clarity, we view each source as having its own separate buffer to queue packets. Here, packets generated by the source wait for access to the channel.

Packet transmissions on the channel are synchronized. More precisely, time is divided into a sequence of fixed-length intervals or time slots. Each time slot is just large enough to allow the transmission of one packet, and packet transmissions must occur within time-slot boundaries. Hence, a packet arriving at the queue, at the very least, must wait until the start of the next time slot before its transmission can begin.

Packets from any given source are served (i.e., transmitted) on a first-come first-serve basis. The sources, however, are assigned fixed priorities: the first source has the highest priority, the last has the lowest. At the start of each time slot, the first packet queued from the highest-priority source is served. That is, a packet at the head of the source k buffer is transmitted if and only if the buffers associated with sources 1 to $k-1$ are empty. Hence, an ongoing packet transmission cannot be preempted; however, an ongoing message transmission will be preempted (at the next slot boundary) whenever there is a message arrival from a higher-priority source.

Such a priority queueing discipline arises naturally in many packet communication systems. The channel might be a link in a data communication network, or may simply be a shared data bus. The use of priority may be required to give more urgent messages lower delay. For example, one might choose to give network control messages higher priority than interactive data messages, which in turn are given higher priority than long file transfers. In some situations, the priority structure is inherent in the mechanism for sharing the channel among the independent messages sources. This is the case with Datakit,[1]

where the module (i.e., the interface between the source and the channel) with the highest address always wins the channel contention. It is also true of some slotted ring systems, where the physical order of sources along the ring imposes a priority ordering for access to the channel.[2]

The first results on queues with preemptive priority appear to be due to White and Christie.[3] Shortly thereafter, others studied the problem using different assumptions about the service time distribution. A comprehensive treatment of some of the early work is given in Jaiswal,[4] and a more up-to-date, but less comprehensive, discussion may be found in Kleinrock.[5] The models examined, however, all assume an "asynchronous" server where service starting times and preemption times are not constrained to certain periodically recurring points. The use of a synchronous service facility in queueing models arises in the context of computer and data communication systems where there is a natural elementary unit of time such as the machine cycle of a processor, or the bit, byte, or packet transmission time on a channel. Many such models are reviewed, and references given, in Kobayashi and Konheim.[6] As we indicated, the model we have selected for study has applications to slotted ring systems, and it is here that one finds analysis of other models similar to ours. The model that seems to come closest is by Konheim and Meister,[2] where the main differences have to do with the arrival process. Konheim and Meister assume discrete arrivals (between slots) of packets, whereas we assume continuous arrivals of messages with each message containing an arbitrarily distributed number of packets. In this way, we are better able to examine message delays in tbe system.

In this paper we analyze the performance of the above preemptive priority queueing system. We begin in Section II by summarizing the queueing model and introducing performance measures that are of interest. In Section III we derive the average in-queue waiting time for each packet in any given source message. From this result we easily obtain the average delay in transporting a message. The corresponding moment-generating functions are derived in the appendix. Finally, in Section IV, we compare performance witb the First-Come First-Served (FCFS) queueing discipline.

II. QUEUEING MODEL

In this section we briefly summarize the important points of the queueing model, and indicate the steady-state statistics that are of interest. Notation established here is used in the performance analysis that follows.

The queueing system under study has the following properties:

1. N sources of messages.

2. Priorities are assigned to sources in decreasing order (i.e., source k has higher priority than source $k + 1$, $k = 1, 2, \cdots, N - 1$).

3. Source k generates messages as an independent Poisson process with rate λ_k messages per time slot. Each such message has its length (in packets) selected independently from the distribution $P_{m_k}(\cdot)$ with first and second moments, $\overline{m}_k$ and $\overline{m_k^2}$, respectively.

4. During busy periods, one packet is transmitted in each time slot and is always selected at the beginning of the time slot from the head of the highest-priority, nonempty source buffer.

5. Each source buffer is assumed infinite, and packets enter and are removed from the buffer on a first-in first-out basis.

We define W_{kj} as the steady-state in-queue waiting time for the jth packet in a message from source k, $k = 1, 2, \cdots, N$. In addition, we define

$$\rho_k = \lambda_k \overline{m}_k,$$

where ρ_k is interpreted as the fraction of time the server is busy with source k packets. We also find it convenient to define

$$\sigma_k = \sum_{i=1}^{k} \rho_i.$$

Other notation is introduced as needed in the analysis.

III. PERFORMANCE ANALYSIS

We begin this section by deriving $\overline{W}_{kj}$, the average in-queue waiting time for the jth packet in a message from source k. Using this result we then obtain the average delay in transporting a message from source k. Included in the discussion are specific numerical examples to illustrate the derived results.

3.1 Average waiting time analysis

In Fig. 2, observe that we may express the waiting time for the jth packet in a source k message as

$$W_{kj} = W_{k1} + \sum_{\ell=1}^{j-1} w_{k\ell}, \tag{1}$$

where the incremental waiting time $w_{k\ell}$ is defined by

$$w_{k\ell} = W_{k,\ell+1} - W_{k\ell}.$$

For a given message length, m_k, the random variables $\{w_{k1}, w_{k2}, \ldots, w_{k,m_k-1}\}$ are independent and identically distributed. We observe that at the beginning of a slot during which a packet from source k is in service, there are no packets from sources 1 to $k - 1$ in the system.

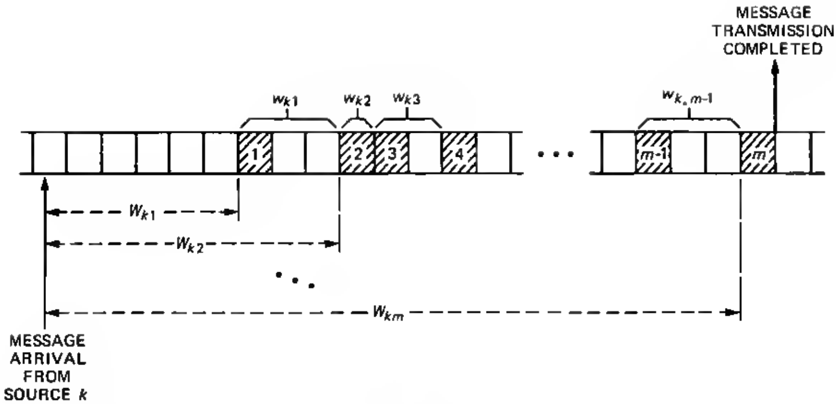

Fig. 2—Waiting times for packet transmissions.

Any messages that arrive from sources 1 to $k-1$ while this source k packet is in service spawn a busy period of service, starting in the next slot, for sources 1 to $k-1$. All such busy periods are independent and identically distributed, and hence so are the random variables $\{w_{k1}, w_{k2}, \ldots, w_{k,m_k-1}\}$.

Note that the incremental waiting time $w_{k\ell}$ consists of one slot time to transmit the ℓth packet in the source k message plus the time to serve all messages from sources 1 to $k-1$ that arrive in the interval $w_{k\ell}$. Hence, the average incremental waiting time $\overline{w}_{k\ell}$ satisfies

$$\overline{w}_{k\ell} = 1 + \sigma_{k-1}\overline{w}_{k\ell},$$

from which we obtain

$$\overline{w}_{k\ell} = \frac{1}{1-\sigma_{k-1}}.$$

It then follows from (1) that the average in-queue waiting time for the jth packet in a source k message is given by

$$\overline{W}_{kj} = \overline{W}_{k1} + \frac{j-1}{1-\sigma_{k-1}}. \tag{2}$$

Hence we are left with having to determine $\overline{W}_{k1}$, the average waiting time for the first packet in the message.

By applying standard queueing arguments, we have

$$\overline{W}_{k1} = \frac{1}{2} + \sum_{i=1}^{k}\sum_{j=1}^{\infty}\rho_{ij}\overline{W}_{ij} + \sum_{i=1}^{k-1}\rho_i\overline{W}_{k1}, \tag{3}$$

where

$$\rho_{ij} \triangleq \lambda_i \Pr[m_i \geq j]. \tag{4}$$

The first term on the right-hand side of (3) is simply the average time between the arrival of a message and the start of the next slot. The second term is, by Little's result, the average number of packets of equal or higher priority awaiting transmission at the moment the message arrives. Finally, the last term corresponds to the average number of packets of higher priority that arrive while the first packet in the source k message waits on queue.

Now substituting (2) into (3) yields

$$\overline{W}_{k1} = \frac{1}{2} + \sum_{i=1}^{k} \sum_{j=1}^{\infty} \rho_{ij} \left(\overline{W}_{i1} + \frac{j-1}{1-\sigma_{i-1}} \right) + \sum_{i=1}^{k-1} \rho_i \overline{W}_{k1}. \tag{5}$$

Note from the definition of ρ_{ij} in (4) that

$$\begin{aligned} \sum_{j=1}^{\infty} \rho_{ij} &= \lambda_i \sum_{j=1}^{\infty} \Pr[m_i \geq j] = \lambda_i \sum_{j=1}^{\infty} \sum_{\ell=j}^{\infty} \Pr[m_i = \ell] \\ &= \lambda_i \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} \Pr[m_i = \ell] = \lambda_i \sum_{\ell=1}^{\infty} \ell \Pr[m_i = \ell] \\ &= \lambda_i \overline{m}_i = \rho_i. \end{aligned} \tag{6}$$

Similarly, we have that

$$\sum_{j=1}^{\infty} (j-1)\rho_{ij} = \frac{\lambda_i}{2} (\overline{m_i^2} - \overline{m}_i). \tag{7}$$

Hence, using (6) and (7), we may rewrite (5) as

$$\overline{W}_{k1} = \frac{\frac{1}{2} + \sum_{i=1}^{k-1} \rho_i \overline{W}_{i1} + \sum_{i=1}^{k} \lambda_i (\overline{m_i^2} - \overline{m}_i)/2(1-\sigma_{i-1})}{(1-\sigma_k)}.$$

Solving recursively, we obtain

$$\overline{W}_{k1} = \frac{1 + \sum_{i=1}^{k} \lambda_i (\overline{m_i^2} - \overline{m}_i)}{2(1-\sigma_k)(1-\sigma_{k-1})}. \tag{8}$$

Finally, substituting (8) into (2) yields

$$\overline{W}_{kj} = \frac{1 + \sum_{i=1}^{k} \lambda_i (\overline{m_i^2} - \overline{m}_i)}{2(1-\sigma_k)(1-\sigma_{k-1})} + \frac{j-1}{1-\sigma_{k-1}}. \tag{9}$$

This concludes the derivation of the average in-queue waiting time $\overline{W}_{kj}$. The derivation of the moment-generating function for W_{kj} (from which $\overline{W}_{kj}$ can be obtained directly) is given in the appendix.

To illustrate the performance, we begin by considering a homoge-

neous system where $\lambda_k = \lambda$, $\overline{m}_k = \overline{m}$, and $\overline{m_k^2} = \overline{m^2}$ for $k = 1, 2, \cdots, N$. For this case, (8) becomes

$$\overline{W}_{k1} = \frac{1 + \frac{k}{N}\rho\,(\overline{m^2}/\overline{m} - 1)}{2\left(1 - \frac{k}{N}\rho\right)\left(1 - \frac{(k-1)}{N}\rho\right)}, \tag{10}$$

where ρ is the total system utilization (or load) defined by

$$\rho = \sum_{i=1}^{N} \rho_i$$

$$= N\lambda\overline{m} \qquad \text{(for a homogeneous system).}$$

If we take $N = 10$ and assume a constant message length of 10 packets (i.e., $\overline{m} = 10$, $\overline{m^2} = 100$), Fig. 3 is a plot of $\overline{W}_{k1}$ vs. ρ for k varying from

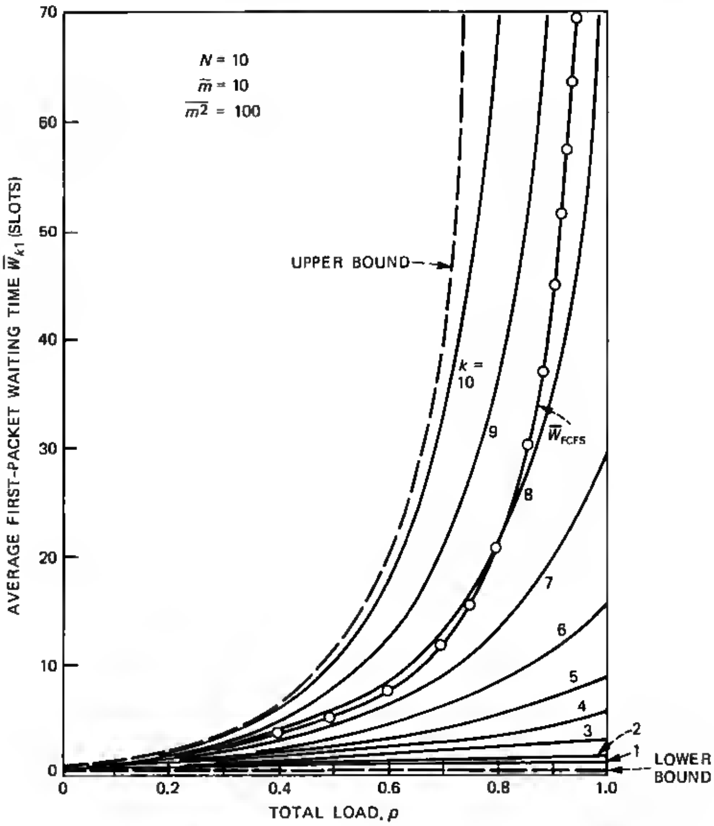

Fig. 3—Average first packet waiting time $\overline{W}_{k1}$ vs. total load ρ.

1 to 10. Also shown in Fig. 3 is the average waiting time for the first-come first-served (FCFS) queueing discipline, which is derived in Section IV. Note from (10) that if we allow $N \to \infty$, then

$$\overline{W}_{11} \to \frac{1}{2}$$

$$\overline{W}_{N1} \to \frac{1 + \rho(\overline{m^2}/\overline{m} - 1)}{2(1 - \rho)^2}.$$

These two expressions represent, respectively, lower and upper hounds on the average first packet waiting time for all sources and arbitrary N. These hounds are plotted as dashed lines in Fig. 3. Finally, if we assume the same values for N, $\overline{m}$, and $\overline{m^2}$ as in Fig. 3, Fig. 4 is a plot of the average incremental waiting time $\overline{w}_{k\ell}$ vs. ρ for k varying from 1 to 10. Also shown in Fig. 4 is the upper hound $1/(1 - \rho)$ on $\overline{w}_{k\ell}$, valid for all parameter values.

3.2 Average message delay analysis

We now consider the average message delay. Defining $\overline{D}_k(m)$ as the average delay (in slots) from the arrival to the queue of an m-packet

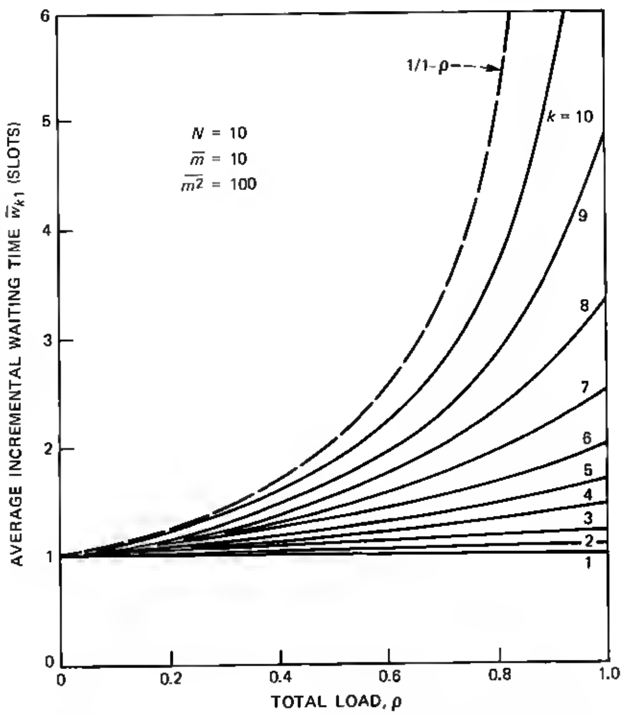

Fig. 4—Average incremental waiting time $\overline{w}_{k1}$ vs. total load ρ.

message from source k until the end of its transmission, we bave

$$\overline{D}_k(m) = \overline{W}_{km} + 1.$$

Letting $\overline{D}_k$ denote the average delay over all messages from source k, it follows, since $\overline{W}_{km}$ is linear in m, that

$$\overline{D}_k = \sum_m \overline{D}_k(m) P_{m_k}(m)$$
$$= \overline{W}_{k,\overline{m}_k} + 1. \tag{11}$$

If we assume the same homogeneous system as represented in Figs. 3 and 4, Fig. 5 is a plot of $\overline{D}_k$ vs. ρ for k varying from 1 to 10. Also shown in Fig. 5 is an upper bound on $\overline{D}_k$, obtained from the upper bounds on $\overline{W}_{k1}$ and $\overline{w}_{k\ell}$. Specifically, we have

$$\overline{D}_k \le \frac{1 + \rho[\overline{m^2}/\overline{m} - 1]}{2(1 - \rho)^2} + \frac{\overline{m} - 1}{(1 - \rho)} + 1,$$

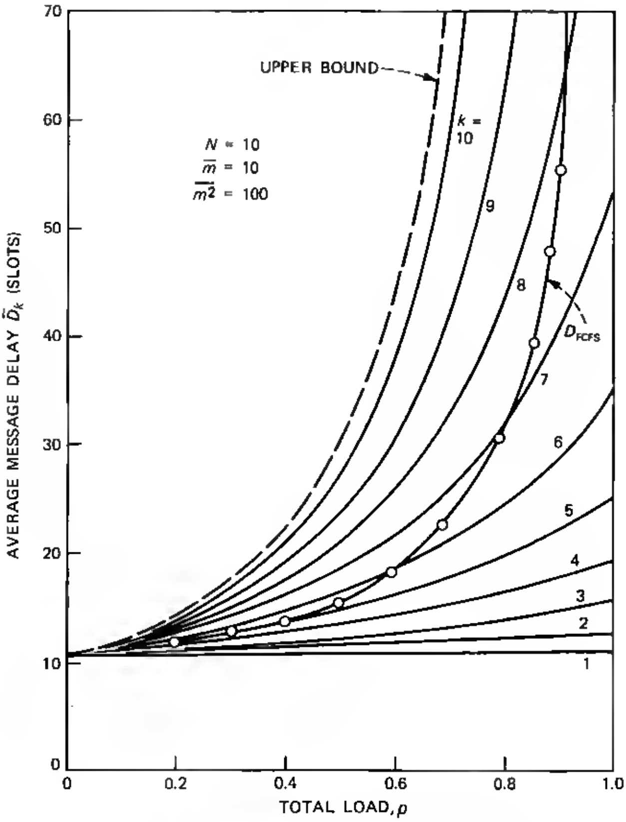

Fig. 5—Average message delay $\overline{D}_k$ vs. total load ρ.

which depends on $\overline{m}$ and $\overline{m^2}$, hut is valid for all sources and arhitrary N.

To complete this section, we consider a nonhomogeneous system consisting of 10 host computers and 300 terminals. The terminals and hosts correspond to the message sources and may he viewed as sharing a common time-slotted bus. There is a priority ordering of the terminals and hosts, with terminals having priority over hosts (i.e., the terminals correspond to sources 1 to 300 and the hosts correspond to sources 301 to 310). Each host is assumed to generate two types of traffic: host-to-host file transfers consisting of fixed-length 32-packet messages, and host-to-terminal messages with an average message length of 2 packets and a standard deviation of 1. Each terminal, on the other hand, only generates messages that are one packet in length and destined to a host. The message generation rates for each of the two types of host traffic are the same for all hosts. Similarly, all terminals generate messages at the same rate. The specific generation rate of each traffic type is such that the total load on the channel is divided as follows: 30 percent host-to-host, 60 percent host-to-terminal, and 10 percent terminal-to-host. The average delay performance for this system is plotted in Fig. 6. Ohserve that the results ohtained allow us to distinguish hetween different types of traffic generated by the same source. In particular, in Fig. 6, the average message delay performance for the host-to-host and host-to-terminal traffic are shown separately.

From the moment-generating function for D_k derived in the appendix, one can ohtain the second moment of the message delay. This, in turn, may he used to compute the message delay standard deviation. For hosts 1 and 10 (i.e., the two extremes), shown in Fig. 7 for the host-to-host messages and in Fig. 8 for the host-to-terminal messages, we see the mean delay and mean delay plus one, two, and three standard deviations (denoted hy 1σ, 2σ, and 3σ). The second-moment-of-message delay depends on the first three moments of message length, and in Fig. 8 we set $\overline{m^3} = 15$.

IV. COMPARISONS WITH FCFS

In this section we compare the average delay performance of the priority queueing discipline studied in the previous section with that of the First-Come First-Served (FCFS) discipline. With the FCFS discipline, messages are served in the order in which they are generated, independent of the source from which they originate. In this way, the FCFS discipline allocates the communication channel more fairly than does the priority discipline. For simplicity, we assume in the analysis a homogeneous system where $\lambda_k = \lambda$, $\overline{m}_k = \overline{m}$, and $\overline{m_k^2} = \overline{m^2}$ for $k = 1, 2, \cdots, N$.

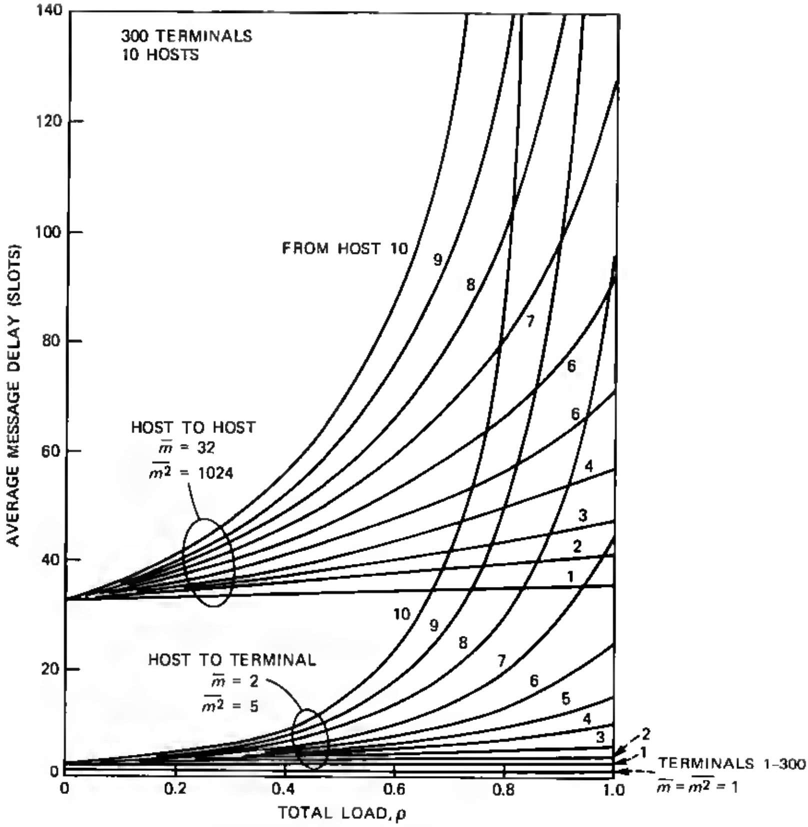

Fig. 6—Average message delay vs. total load ρ.

The performance analysis of the FCFS queueing discipline is a special case of the results obtained for the priority discipline. Specifically, we combine the N independent Poisson streams into a single Poisson stream (using the well-known result that the sum of independent Poisson processes is a Poisson process) with rate $N\lambda$. From (10) we have that the average in-queue waiting time for a message generated by this combined (single) source is given by

$$\overline{W}_{\text{FCFS}} = \frac{\rho(\overline{m^2}/\overline{m})}{2(1-\rho)} + \frac{1}{2}, \tag{12}$$

where again $\rho = N\lambda\overline{m}$ is the total system utilization. The average message delay for the FCFS system is given by

$$\begin{aligned}\overline{D}_{\text{FCFS}} &= \overline{W}_{\text{FCFS}} + \overline{m} \\ &= \frac{\rho(\overline{m^2}/\overline{m})}{2(1-\rho)} + \frac{1}{2} + \overline{m}. \end{aligned}\tag{13}$$

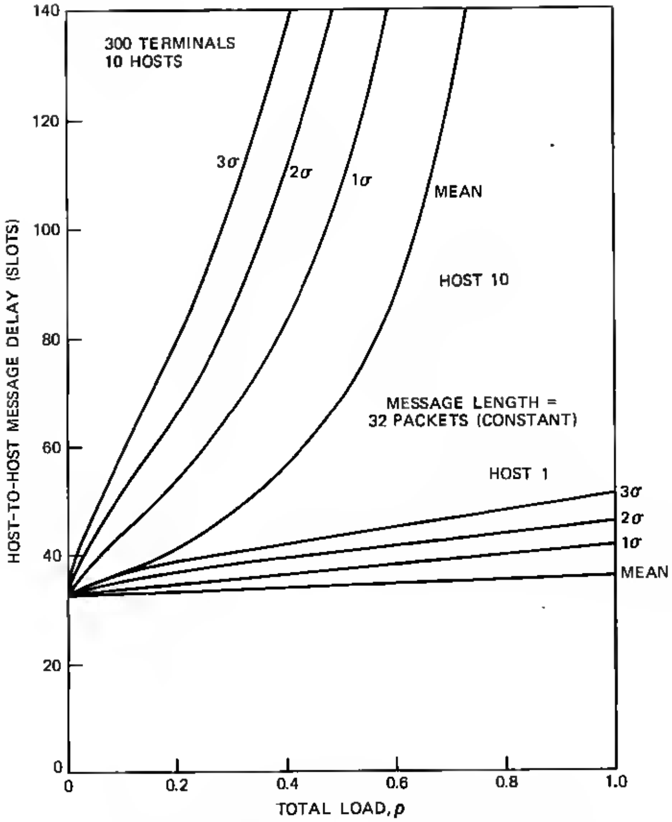

Fig. 7—Host-to-host message delay vs. total load ρ.

$\overline{W}_{\text{FCFS}}$ is plotted in Fig. 3 and $\overline{D}_{\text{FCFS}}$ is plotted in Fig. 5 for the assumed system parameter values.

It is worth noting that the waiting time and delay results given by (12) and (13), respectively, differ from those corresponding to the standard M/G/1 queueing system by the additional term 1/2. This added term results from the synchronous nature of tbe server and represents the average time an arriving message must wait before the start of the next time slot.

We continue the priority and FCFS comparison by focusing on the unfairness issue. Specifically, we consider the ratio of the average message delay for source N to that of source 1, $\overline{D}_N/\overline{D}_1$. Since all sources encounter the same average delay in the FCFS discipline, $\overline{D}_N/\overline{D}_1 = 1$. With the priority discipline, source N has the lowest priority and source 1 the highest; hence $\overline{D}_N/\overline{D}_1 > 1$ for $\rho > 0$. In particular, we have from (9) and (11) that

$$\overline{D}_k = \frac{1 + \frac{k}{N}\rho[(1 + c_m^2)\overline{m} - 1]}{2\left(1 - \frac{k}{N}\rho\right)\left(1 - \frac{(k-1)}{N}\rho\right)} + \frac{\overline{m} - 1}{\left(1 - \frac{(k-1)}{N}\rho\right)} + 1, \quad (14)$$

where c_m^2 is the squared coefficient of variation for the message-length distribution defined by

$$c_m^2 = \frac{\text{variance}(m)}{(\overline{m})^2}.$$

Hence, for large N we have from (14) that

$$\begin{aligned}\overline{D}_1 &= \frac{\frac{\rho}{N}(1 + c_m^2)\overline{m}}{2\left(1 - \frac{\rho}{N}\right)} + \frac{1}{2} + \overline{m} \\ &\approx \frac{\overline{m}}{2}\left[\frac{1}{\overline{m}} + \frac{\rho}{N}(1 + c_m^2) + 2\right]\end{aligned}$$

and

$$\begin{aligned}\overline{D}_N &= \frac{1 + \rho[(1 + c_m^2)\overline{m} - 1]}{2(1 - \rho)\left(1 - \frac{(N-1)}{N}\rho\right)} + \frac{\overline{m} - 1}{\left(1 - \frac{(N-1)}{N}\rho\right)} + 1 \\ &\approx \frac{\overline{m}\left[(1 + c_m^2)\rho + \left(2 - \frac{1}{\overline{m}}\right)(1 - \rho) + \frac{2}{\overline{m}}(1 - \rho)^2\right]}{2(1 - \rho)^2}.\end{aligned}$$

It follows then that

$$\frac{\overline{D}_N}{\overline{D}_1} \approx \begin{cases} \dfrac{1 + c_m^2\rho + 2(1 - \rho)^2}{3(1 - \rho)^2} & \text{for} \quad \overline{m} = 1 \\[2ex] \dfrac{2 + (c_m^2 - 1)\rho}{2(1 - \rho)^2} & \text{for} \quad \overline{m} \gg 1. \end{cases}$$

Observe that for large N and fixed ρ, the increase in $\overline{D}_N/\overline{D}_1$ is approximately linear with the squared coefficient of variation c_m^2. In Fig. 9, the ratio $\overline{D}_N/\overline{D}_1$ is plotted against total utilization ρ for tbe FCFS and priority disciplines with $c_m^2 = 0$ and 1.

To complete this section, we compare tbe average delay performance of the FCFS discipline with tbe overall average delay of the priority discipline. That is, we compare $\overline{D}_{\text{FCFS}}$ as given by (13) to the quantity

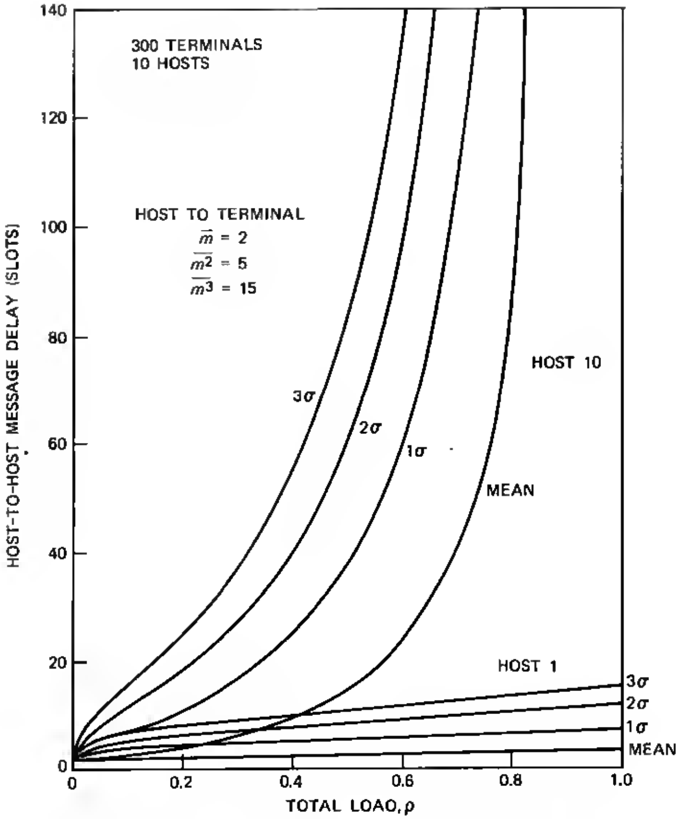

Fig. 8—Host-to-terminal message delay vs. total load ρ.

$$\overline{D} \triangleq \frac{1}{N} \sum_{k=1}^{N} \overline{D}_k.$$

Using the expression for $\overline{D}_k$ given in (14), we obtain after some manipulation

$$\overline{D} = \frac{(1 + c_m^2)\overline{m}}{2(1 - \rho)} + \frac{1}{2N} [2\overline{m} - 1 - (1 + c_m^2)\overline{m}]\gamma + 1,$$

where

$$\gamma = \sum_{k=1}^{N} \left[1 - \frac{(k-1)}{N} \rho\right]^{-1}.$$

From this, one may show that

$$\overline{D}_{\text{FCFS}} \leq \overline{D} \quad \text{for} \quad 0 \leq c_m^2 \leq \frac{\overline{m} - 1}{\overline{m}}$$

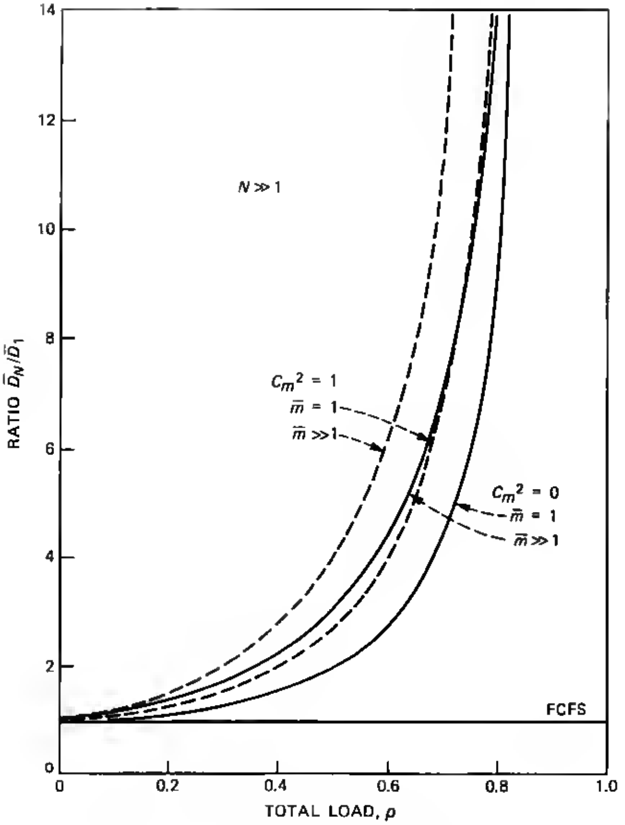

Fig. 9—Ratio $\overline{D}_N/\overline{D}_1$ vs. total load ρ.

and

$$\overline{D}_{\text{FCFS}} \geq \overline{D} \quad \text{for} \quad c_m^2 \geq \frac{\overline{m} - 1}{\overline{m}}.$$

Hence, for sufficiently large message-length coefficient of variation c_m^2, the overall average delay for the priority discipline is less than the average delay for the first-come first-served discipline. Of course, as we saw earlier, as c_m^2 increases so does the relative unfairness of the priority discipline over the FCFS discipline.

V. CONCLUSIONS

We analyzed the performance of a preemptive priority queue, which has direct applications to packet communication systems. The main distinguishing feature of the system studied compared to the standard M/G/1 preemptive resume priority queue[5] is that the server can only

begin serving a "customer" (and preemptions take place) at integer multiples of time corresponding to packet slot boundaries in the communication context. Mean value formulas for in-queueing waiting time and average message delay were derived and comparisons made to the FCFS queueing discipline. A derivation of the waiting time and delay moment-generating functions is given in the appendix.

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APPENDIX

Derivation of the Waiting Time and Message Delay Moment-Generating Functions

As we introduced in Section II, W_{kj} is the steady-state in-queue waiting time for the jth packet in a message from source k, $k = 1, 2, \cdots, N$. Its Moment-Generating Function (MGF), defined as

$$G_{W_{kj}}(\nu) = E[e^{\nu W_{kj}}]$$

is derived in this appendix. From this result, the message delay MGF is easily obtained. The approach taken parallels in many respects tbe analysis given in Section III.

We begin the derivation by examining the duration of a busy period for sources 1 to $k - 1$, denoted by Y_k. Such a busy period starting in a slot is initialized by one or more message arrivals from sources 1 to $k - 1$ in the previous slot (which contains no packet from sources 1 to $k - 1$). Let A_k denote the total number of packets that arrive from sources 1 to $k - 1$ in this previous slot. For the ith packet in this set, we define the "sub-busy" period $X_k(i)$ to consist of the duration of the "virtual" busy period (i.e., as if $i = A_k = 1$) initiated by the messages (if any) that arrive from sources 1 to $k - 1$ while this ith packet is in service. In other words, we conceptually reorder the priorities so that each of the A_k packets, and the sub-busy period it spawns, is served in turn. This does not change Y_k and is a standard approach to busy-period analysis.

Due to the memoryless property of the arrival process, the sub-busy period random variables $X_k(i)$, $i = 1, 2, \cdots, A_k$, are independent and

identically distributed (*iid*). In addition, note that Y_k has the same distribution as the generic random variable X_k, and satisifies the relation

$$Y_k = A_k + \sum_{i=1}^{A_k} X_k(i). \tag{15}$$

The Probability-Generating Function (PGF) for the discrete random variable X_k is defined as

$$\Phi_{X_k}(z) = E[z^{X_k}] = \sum_{i=0}^{\infty} z^i \Pr[X_k = i].$$

Using (15) and the result that X_k is distributed as Y_k, we obtain

$$\begin{aligned} \Phi_{X_k}(z) &= E[z^{Y_k}] \\ &= E\left[z^{A_k + \sum_{i=1}^{A_k} X_k(i)}\right] \\ &= E[(z\Phi_{X_k}(z))^{A_k}] \\ &= \Phi_{A_k}(z\Phi_{X_k}(z)), \end{aligned} \tag{16}$$

where $\Phi_{A_k}(z)$ is the PGF for the random variable A_k.

Now, A_k is equal to the total number of packets arriving from sources 1 to $k-1$ in one time slot. Recall that each source i generates messages as an independent Poisson process with rate λ_i; and each such message has its length selected independently from the distribution $P_{m_i}(\cdot)$, whose PGF we denote by $\Phi_{m_i}(z)$. It follows then that

$$\begin{aligned} \Phi_{A_k}(z) &= \prod_{i=1}^{k-1} \left\{ \sum_{r=0}^{\infty} \frac{\lambda_i^r e^{-\lambda_i}}{r!} \cdot [\Phi_{m_i}(z)]^r \right\} \\ &= \prod_{i=1}^{k-1} e^{\lambda_i[\Phi_{m_i}(z)-1]}. \end{aligned} \tag{17}$$

Hence, substituting (17) into (16), we obtain

$$\Phi_{X_k}(z) = \prod_{i=1}^{k-1} e^{\lambda_i[\Phi_{m_i}(z\Phi_{X_k}(z))-1]}. \tag{18}$$

As we shall see, $\Phi_{X_k}(z)$, the PGF for the duration of a busy period for sources 1 to $k-1$, plays an important role in the derivation of $G_{W_{kj}}(v)$, the waiting time MGF.

Returning to eq. (1) in Section III, we note that W_{kj} is the sum of W_{k1} and the $j-1$ *iid* random variables $w_{k1}, w_{k2}, \cdots, w_{k,j-1}$. Observe, however, that $w_{k\ell}$, $\ell = 1, 2, \cdots, j-1$, is distributed as $X_k + 1$. That is, $w_{k\ell}$ is composed of the service time for the ℓth packet plus the busy

period for sources 1 to $k - 1$ initiated during this service time. In addition, it follows that the waiting time for the first packet in a source k message, W_{k1}, is statistically independent of $w_{k\ell}$, $\ell = 1, 2, \cdots, j-1$. Hence we may write

$$G_{W_{kj}}(\nu) = G_{W_{k1}}(\nu)\cdot[e^{\nu}\Phi_{X_k}(e^{\nu})]^{j-1}. \tag{19}$$

This leaves us with having to determine the MGF for W_{k1}.

Let us for the moment consider the time-dependent behavior for the number of packets queued from sources 1 to k. For time slot n, we let $Q_k(n)$ denote the number of such packets queued just after the beginning of the slot and, to be consistent with our previous notation, we let $A_{k+1}(n)$ denote the number of packets that arrive from sources 1 to k during the nth slot. It follows that

$$Q_k(n+1) = [Q_k(n) + A_{k+1}(n) - 1]^+, \tag{20}$$

where

$$[\epsilon]^+ = \begin{cases} \epsilon & \text{if} \quad \epsilon \ge 0 \\ 0 & \text{if} \quad \epsilon < 0. \end{cases}$$

From (20) we obtain the relation

$$E[z^{Q_k(n+1)}] = E[z^{[Q_k(n)+A_{k+1}(n)-1]^+}], \tag{21}$$

which may be rewritten as

$$\begin{aligned}
\Phi_{Q_k(n+1)}(z) &= E[z^{[Q_k(n)+A_{k+1}-1]^+}] \\
&= \Pr[Q_k(n) + A_{k+1} = 0] + \Pr[Q_k(n) + A_{k+1} > 0] \\
&\quad \cdot z^{-1}E[z^{Q_k(n)+A_{k+1}} \mid Q_k(n) + A_{k+1} > 0] \\
&= \Pr[Q_k(n) = 0]\Pr[A_{k+1} = 0] \\
&\quad + z^{-1} \sum_{i=1}^{\infty} z^i \Pr[Q_k(n) + A_{k+1} = i] \\
&= \Pr[Q_k(n) = 0]\Pr[A_{k+1} = 0] \\
&\quad + z^{-1}\{E[z^{Q_k(n)+A_{k+1}}] - \Pr[Q_k(n) = 0]\Pr[A_{k+1} = 0]\} \\
&= \Pr[Q_k(n) = 0]\Pr[A_{k+1} = 0](1 - z^{-1}) \\
&\quad + z^{-1}\Phi_{Q_k(n)}(z)\Phi_{A_{k+1}}(z),
\end{aligned} \tag{22}$$

where we have used the fact that $Q_k(n)$ and A_{k+1} are statistically independent. Taking the limit as $n \to \infty$ on both sides of (22) (the limits exist for $\sigma_k < 1$) yields

$$\Phi_{Q_k}(z) = \Pr[Q_k = 0]\Pr[A_{k+1} = 0](1 - z^{-1}) + z^{-1}\Phi_{Q_k}(z)\Phi_{A_{k+1}}(z), \tag{23}$$

where Q_k represents the steady-state number of packets queued from sources 1 to k at the beginning of a slot. Rearranging the terms in (23), we obtain

$$\Phi_{Q_k}(z) = \frac{\Pr[Q_k = 0]\Pr[A_{k+1} = 0](z - 1)}{z - \Phi_{A_{k+1}}(z)}. \tag{24}$$

Taking the limit as $z \longrightarrow 1$ on both sides of (24) yields

$$\Pr[Q_k = 0]\Pr[A_{k+1} = 0] = 1 - \frac{\partial}{\partial z} \Phi_{A_{k+1}}(z)\bigg|_{z=1}$$

$$= 1 - \sigma_k.$$

Hence, using this result and (17), (24) may be rewritten as

$$\Phi_{Q_k}(z) = \frac{(1 - \sigma_k)(z - 1)}{z - \prod_{i=1}^{k} e^{\lambda_i[\Phi_{m_i}(z) - 1]}}. \tag{25}$$

Now consider the end of the time slot during which a source k message is generated. The number of packets of higher or equal priority that are queued and must be transmitted before the first packet in this source k message is given by

$$Q_k + A_k + B_k,$$

where Q_k is the number of queued packets from sources 1 to k just after the beginning of the slot, A_k is the number of packets from sources 1 to $k - 1$ that arrive during the slot, and the new random variable, B_k, represents the number of packets from source k that arrive during the slot prior to the generation of the source k message in question. The ith packet in this set of $(Q_k + A_k + B_k)$ packets initiates a sub-busy period of duration $X_k(i)$. Hence we may write

$$W_{k1} = U + \sum_{i=0}^{(Q_k+A_k+B_k)} [1 + X_k(i)], \tag{26}$$

where U is a random variable, uniformly distributed over one slot time, that represents the time from when the source k message is generated until the start of the next slot.

From (26) we may write

$$E[e^{\nu W_{k1}} \mid U = u,\ Q_k = q_k,\ A_k = a_k,\ B_k = b_k] = e^{\nu u}[e^{\nu}\Phi_{X_k}(e^{\nu})]^{(q_k+a_k+b_k)}.$$

Removing the conditioning on the independent random varaibles Q_k and A_k yields

$$E[\alpha^{W_{k1}} \mid U = u,\ B_k = b_k]$$

$$= \alpha^u[\alpha\Phi_{X_k}(\alpha)]^{b_k}\Phi_{Q_k}(\alpha\Phi_{X_k}(\alpha))\Phi_{A_k}(\alpha\Phi_{X_k}(\alpha)), \tag{27}$$

where, for simplicity, we have substituted α for e^{ν}. Now, using the same approach as we did witb A_k, we obtain

$$E[z^{B_k} | U = u] = e^{\lambda_k(1-u)[\Phi_{m_k}(z)-1]}.$$

Thus, removing the conditioning on B_k in (27) yields

$$E[\alpha^{W_{k1}} | U = u] = \alpha^u e^{\lambda_k(1-u)[\Phi_{m_k}(\alpha\Phi_{X_k}(\alpha))-1]} \cdot \Phi_{Q_k}(\alpha\Phi_{X_k}(\alpha))\Phi_{A_k}(\alpha\Phi_{X_k}(\alpha))$$

$$= [e^{\nu-\lambda_k[\Phi_{m_k}(\alpha\Phi_{X_k}(\alpha))-1]}]^u \Phi_{Q_k}(\alpha\Phi_{X_k}(\alpha))\Phi_{A_{k+1}}(\alpha\Phi_{X_k}(\alpha)).$$

Now, removing the conditioning on U, we obtain

$$G_{W_{k1}}(\nu) = G_U\{\nu - \lambda_k[\Phi_{m_k}(\alpha\Phi_{X_k}(\alpha)) - 1]\}\Phi_{Q_k}(\alpha\Phi_{X_k}(\alpha))\Phi_{A_{k+1}}(\alpha\Phi_{X_k}(\alpha)),$$

where

$$G_U(\nu) = \int_0^1 e^{\nu u} du = \frac{1}{\nu}[e^{\nu} - 1]. \tag{28}$$

Finally using (19), we obtain

$$G_{W_{kj}}(\nu) = G_U\{\nu - \lambda_k[\Phi_{m_k}(\alpha\Phi_{X_k}(\alpha)) - 1]\}$$
$$\cdot \Phi_{Q_k}(\alpha\Phi_{X_k}(\alpha))\Phi_{A_{k+1}}(\alpha\Phi_{X_k}(\alpha)) \cdot [\alpha\Phi_{X_k}(\alpha)]^{j-1},$$

where $\alpha = e^{\nu}$, $G_U(\nu)$ is given by (28), $\Phi_{Q_k}(z)$ is given by (25), $\Phi_{A_{k+1}}(z)$ is given by (17), and $\Phi_{X_k}(z)$ is given by (18).

The delay in transmitting a source k message of length m, $D_k(m)$, is given by

$$D_k(m) = W_{km} + 1.$$

Hence, the MGF for $D_k(m)$ is given by

$$G_{D_k(m)}(\nu) = e^{\nu} G_{W_{km}}(\nu).$$

It follows that the MGF for D_k, the delay in transmitting a randomly selected source k message, is given by

$$G_{D_k}(\nu) = G_{W_{k1}}(\nu)\Phi_{m_k}(e^{\nu}\Phi_{X_k}(e^{\nu}))/\Phi_{X_k}(e^{\nu}).$$

AUTHORS

Robert R. Boorstyn, B.E.E., 1958, City College of New York; M.S. (Electrical Engineering), and Pb.D. (Electrical Engineering), 1963 and 1966, respectively, Polytechnic Institute of Brooklyn; Sperry Gyroscope Company, 1958–1961; Polytechnic Institute of New York, 1961—; on leave at Bell Laboratories, 1977–1978; consultant at Bell Laboratories, 1981–1982. Mr. Boorstyn is engaged in research on computer communication networks, specializing in packet radio networks, routing, network design, and analysis. He is currently a Professor of Electrical Engineering and Computer Science at Polytechnic Institute of New York. He has been Editor for Computer Communications of the *IEEE Transactions on Communications*, Chairman of the Computer Com-

munications Committee of the IEEE Communications Society, and Secretary of the IEEE Information Theory Group. He is an Associate Editor of *Networks Journal.*

Michael G. Hluchyj, B.S. (Electrical Engineering), 1976, University of Massachusetts at Amherst; S.M., E.E., and Ph.D. (Electrical Engineering), Massachusetts Institute of Technology, 1978, 1978, and 1981, respectively; Bell Laboratories, 1981–1982; AT&T Information Systems, 1983—. Mr. Hluchyj's work at Bell Laboratories and AT&T Information Systems has centered around the architectural design and performance analysis of local area data communication networks.

Chan David Tsao, B.S. (Control Engineering), 1973, National Chiao Tung University (Taiwan); M.S. (Systems Engineering), 1978, Florida Institute of Technology; M.S. and Ph.D. (Electrical Engineering), Polytechnic Institute of New York, 1979 and 1982, respectively; Bell Laboratories, 1981–1982; AT&T Information Systems, 1983—. At Bell Laboratories, Mr. Tsao worked on performance evaluation and system design for local area networks. His current research interest is architectures and applications for local area networks.